Approximation Algorithms

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Is a given natural number a prime number?
A computable problem.

Optimization Problems

Optimaization Problem

An *optimization problem* Π consists of a set of instances $\mathcal{I}$, a set of solutions $\mathcal{O}$, and three functions sol: $\mathcal{I} \rightarrow \mathcal{P}(\mathcal{O})$, quality: $\mathcal{I} \times \mathcal{O} \rightarrow \mathbb{R}$, and goal $\in$ *min*, *max*.

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For every instance $I \in \mathcal{I}$ and every feasible solution $O \in sol(I)$, quality (I, O) denotes the measure of I and O. An *optimal solution* for an instance $I \in \mathcal{I}$ of Π is a solution $OPT(I) \in sol(I)$ such that
quality $(I, OPT(I)) =$ goal $\{$ quality $(I, O) \mid O \in sol(I)\}$.

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quality $(I, OPT(I)) =$ goal $\{$ quality $(I, O) \mid O \in sol(I)\}$.
If goal = min, we call Π a *minimization problem* and write “cost” instead of “quality.” Conversely, if goal=max, we say that Π is a *maximization problem* and write “gain” instead of “quality.”

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All those problems are NP-hard problems.
It is unlikely to have polynomial time algorithms.

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Vertex Cover Problem: An Approximation

Algorithm VertexCoverApprox(G, C)

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return $C$
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For $r \geq 1$, ALG is an *r-approximation algorithm* for Π if, for every $I \in \mathcal{I}$,
gain (OPT (I))$\leq r$. gain (ALG (I)) if Π is maximization problem, or
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The *approximation ratio* of ALG is defined as
$r_{ALG} := inf\{r \geq 1 \mid \text{ALG is an } r\text{- approximation algorithm for } \Pi\}$.

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An instance I is given by a sequence of $n+1$ positive integers $B, w_1, w_2, \cdots, w_n$, where we consider w_i with $1 \leq i \leq n$ to be the weight of the ith object;
B is the capacity of the knapsack.

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The gain of a solution $\mathcal{O}$ and a corresponding instance I is given by
gain $(I, \mathcal{O}) = \sum_{i \in O} w_i$.
The goal is to maximize this number.

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Approximation Algorith (Simple Knapsack Problem)

Theorem

KNGREEDY is a polynomial-time 2-approximation algorithm for the simple knapsack problem.

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Proof.

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  - **Case 2.1.** Suppose w_i of weight at least $B/2$. $w_1 \geq B/2$ and w_1 is always packed into knapsack. Since B is an upper bound for any solution, the approximation ratio of KNGREEDY is at most 2 in this case.

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  - **Case 2.2.** Suppose weight of all objects are smaller than $B/2$, j be the index of the first object that is too heavy to be packed into the knapsack by KNGREEDY.
    $w_j < B/2$
    this implies that space that is already occupied by the objects $w_1, w_2, \cdots, w_{j-1}$ must be larger than $B/2$. the approximation ratio of KNGREEDY is at most 2 in this case.

Chapter 1 (Simple Knapsack Problem)

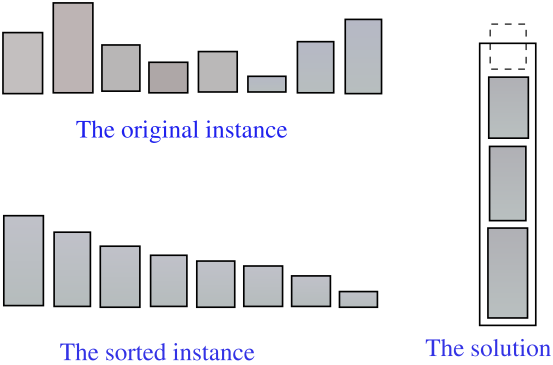

Figure: The greedy strategy; first sort, then pack greedily what fits.

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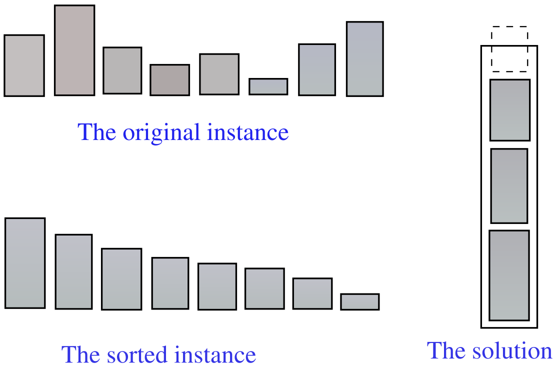

The original instance

The sorted instance

The solution

Figure: The greedy strategy; first sort, then pack greedily what fits.

Chapter 1 (Simple Knapsack Problem)

$$r_{KNGREEDY} \geq \frac{gain(OPT(I))}{gain(KNGREEDY(I))} = \frac{B}{\frac{B}{2}+1} = \frac{2}{1+\frac{2}{B}}$$

Approximability

APX

The class APX (an abbreviation of ”approximable”) is the set of NP optimization problems that allow polynomial-time approximation algorithms with approximation ratio bounded by a constant (or constant-factor approximation algorithms for short). In simple terms, problems in this class have efficient algorithms that can find an answer within some fixed multiplicative factor of the optimal answer.

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PTAS

If there is a polynomial-time δ-approximation algorithms with $\delta = 1 + \epsilon$, for any fixed value $\epsilon > 0$ to solve a problem, then the problem is said to have a polynomial-time approximation scheme (PTAS). The running time depends on input size and ϵ. Unless P=NP there exist problems that are in APX but without a PTAS, so the class of problems with a PTAS is strictly contained in APX

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Fully polynomial-time approximation scheme (FPTAS)

An algorithm that achieves an arbitrarily good approximation ratio of $1 + \epsilon$ in a time that is polynomial both in *n* and $1/\epsilon$ is called a *fully polynomial-time approximation scheme (FPTAS)*.

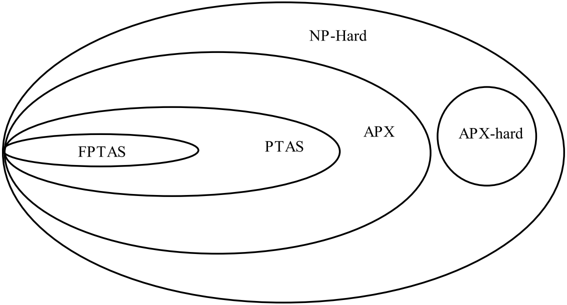
Hard for Approximation

APX-Hard

A problem is said to be APX-hard if there is a PTAS reduction from every problem in APX to that problem, and to be APX-complete if the problem is APX-hard and also in APX. As a consequence of $P \neq NP \Rightarrow PTAS \neq APX$, if $P \neq NP$ is assumed, no APX-hard problem has a PTAS.

TSP problem is APX-hard.

NP-hard, APX-hard, APX, PTAS and FPTAS



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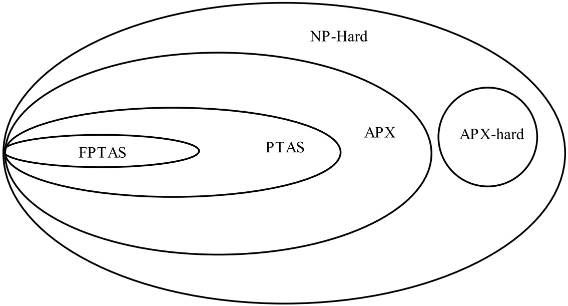

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Approximation for TSP Satisfying Triangle Inequality

On whiteboard.