Implicit Representation of Graphs

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Outline

Implicit Representation of Graphs

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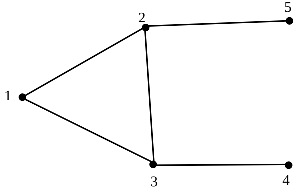
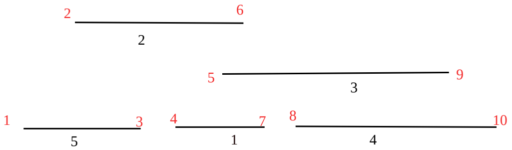
An Implicit Representation of an Interval Graph.





1: 4, 7
2: 2, 6
3: 5, 9
4: 8,10
5: 1, 3

Space optimal Representation and Implicit Representation

Is the implicit representation of an interval graph space optimal?

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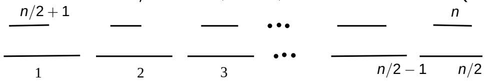

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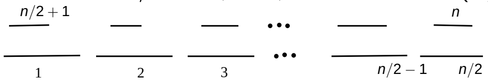

$$(n/2)! = (n/2)(n/2-1)\cdots(n/4)(n/4-1)\cdots 1 > (n/2)(n/2-1)\cdots(n/4+1)(n/4) > (n/4)^{n/4} = 2^{(n/4)log(n/4)} = 2^{\Omega(nlogn)}$$

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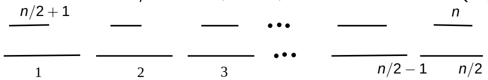

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Representation of all interval graphs requires $\Omega(nlogn)$ bits.

We achieve $O(nlogn)$ bits with the implicit representation.

Thus implicit representation is space optimal.

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- There are classes of graphs which have $2^{\Theta(n^2)}$ members. Any representation of graphs of such classes must use $\Omega(n^2)$ bits for the average graphs in the class, while an implicit representation uses $O(nlogn)$ bits.
- There are graph classes with $2^{O(nlogn)}$ members which do not have implicit representation. Example: 7-sparse graph.

Outline

Implicit Representation of Trees

A graph of n vertices is a *7-sparse graph* if it contains at most $7n$ edges.

Claim: The 7-sparse graphs do not have an implicit representation.

Proof. Suppose that the 7-sparse graphs have an implicit representation. Consider the set of all graphs on n vertices; we know that there are $2^{\Theta(n^2)}$ such graphs. Let G be an arbitrary graph of n vertices. Add n^2 isolated vertices to G, and we have a graph G' which is 7-sparse. G' must have an implicit representation. Adjacencies between vertices in an implicit representation can be determined using only the bits stored at the vertices, so G can be reconstructed from the bits stored at each vertex of G within the representation of G'.

Implicit Representation of Trees

There are $O(log(n^2 + n)$ bits stored at each vertex of G'; thus G can be reconstructed from a set of $O(nlogn)$ bits. This implies that there are $2^{O(nlogn)}$ graphs on n vertices, which is certainly false. So there can not be an implicit representation of the 7-sparse graphs.

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- Therefore, there are $2^{O(nlogn)}$ trees of n vertices.
- But adjacency list representation does not imply implicit representation, since a tree can have vertices of arbitrarily high degree.

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Store only the parent of each vertex v at v.
Two vertices v and w are adjacent if and only if v is the parent of w or w is the parent of v.

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Implicit Representation of Planar Graphs

Every planar graph has a vertex of degree at most 5.

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Every planar graph has a vertex of degree at most 5.
Simple implicit representaton:
Repeatedly remove vertices of degree at most 5 and store all remaining neighbors of the vertex with the vertex.

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Suppose a hereditary graph class have at most cn edges. Then the total size of the adjacency list is $2cn$. This implies that there must be some vertex with degree atmost $2c$ neighbors. We remove this vertex and store all neighbors with this vertex.

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Hereditary Graphs

Reference: Jeremy P. Spinard, Efficient Graph Representations, American Mathematical Society, 2003.