Canonical Decomposition

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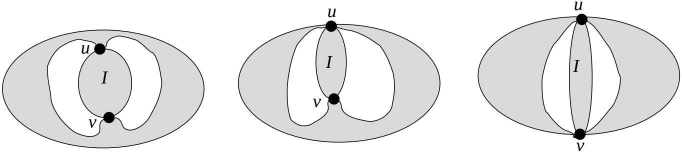
Internally 3-connected Graph

A plane graph G is *internally 3-connected* if G is 2-connected and, for any separation pair $\{u, v\}$ of G, u and v are outer vertices and each connected component of $G - \{u, v\}$ contains an outer vertex.

In other words, G is internally 3-connected if and only if it can be extended to a 3-connected graph by adding a vertex in an outer face and connecting it to all outer vertices.

Internally 3-connected Graph

If a 2-connected plane graph G is not internally 3-connected, then G has a separation pair $\{u, v\}$ of one of the following three types



Chord Path

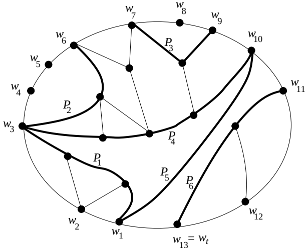
We call a path P in G a *chord-path* of the cycle $C_o(G)$ if P satisfies the following (i)–(iv):

(i) P connects two outer vertices w_p and w_q, $p < q$;

(ii) $\{w_p, w_q\}$ is a separation pair of G;

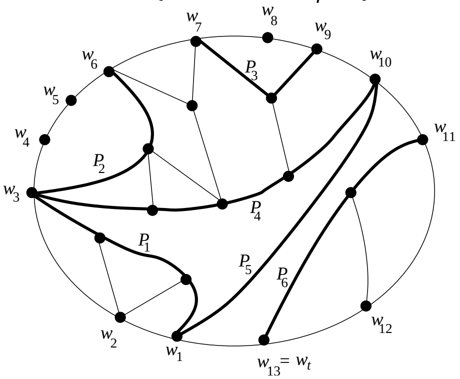
(iii) P lies on an inner face; and

(iv) P does not pass through any outer edge and any outer vertex other than the ends w_p and w_q.



Outer Chain

Let $\{v_1, v_2, \cdots, v_p\}$, $p \geq 3$, be a set of three or more outer vertices consecutive on $C_o(G)$ such that $d(v_1) \geq 3$, $d(v_2) = d(v_3) = \cdots = d(v_{p-1}) = 2$, and $d(v_p) \geq 3$. Then we call the set $\{v_2, v_3, \cdots, v_{p-1}\}$ an *outer chain* of G.



Canonical Decomposition

G_k and $\overline{G_k}$

Let $G = (V, E)$ be a 3-connected plane graph of $n \geq 4$ vertices. For an ordered partition $\Pi = (U_1, U_2, \cdots, U_l)$ of set V, we denote by G_k, $1 \leq k \leq l$, the subgraph of G induced by $U_1 \cup U_2 \cup \cdots \cup U_k$, while we denote by $\overline{G_k}$, $0 \leq k \leq l-1$, the subgraph of G induced by $U_{k+1} \cup U_{k+2} \cup \cdots \cup U_l$. Clearly $\overline{G_k} = G - U_{k+1} \cup U_{k+2} \cdots \cup U_l$, and $G = G_l = \overline{G_0}$. Let (v_1, v_2) be an outer edge of G.

Canonical Decomposition

Let (v_1, v_2) be an outer edge of G. We then say that Π is a *canonical decomposition* of G (for an outer edge (v_1, v_2)) if Π satisfies the following conditions (cd1)–(cd3).

(cd1) U_1 is the set of all vertices on the inner face containing edge (v_1, v_2), and U_l is a singleton set containing an outer vertex $v_n \notin \{v_1, v_2\}$.

(cd2) For each index k, $1 \leq k \leq l$, G_k is internally 3-connected.

(cd3) For each index k, $2 \leq k \leq l$, all vertices in U_k are outer vertices of G_k and the following conditions hold:

- (a) if $|U_k| = 1$, then the vertex in U_k has two or more neighbors in G_{k-1} and has at least one neighbor in $\overline{G_k}$ when $k < l$; and
- (b) If $|U_k| \geq 2$, then U_k is an outer chain of G_k, and each vertex in U_k has at least one neighbor in $\overline{G_k}$.

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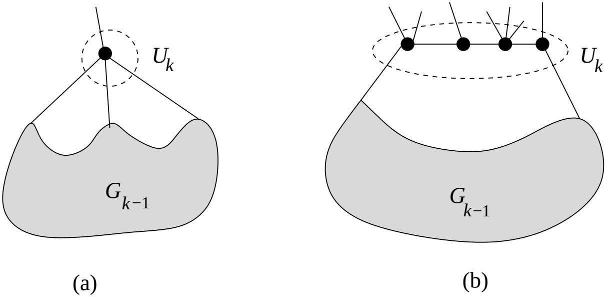

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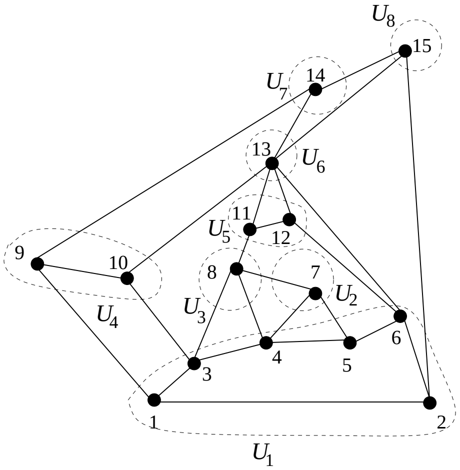

Canonical Decomposition

Lemma Every 3-connected plane graph G of $n \geq 4$ vertices has a canonical decomposition Π, and Π can be found in linear time.

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Lemma Every 3-connected plane graph G of $n \geq 4$ vertices has a canonical decomposition Π, and Π can be found in linear time.

Proof We first show that G has a canonical decomposition. Let U_1 be the set of all vertices on the inner face containing edge (v_1, v_2). Since G is 3-connected and $n \geq 4$, there is an outer vertex $v_n \notin U_1$. We choose the singleton set $\{v_n\}$ as U_l. Thus (cd1) holds. Since $G_l = G$, (cd2) holds for $k = l$. Since G is 3-connected and v_n is on $C_o(G)$, $G_{l-1} = G - v_n$ is internally 3-connected and hence (cd2) holds for $k = l - 1$. Since v_n has degree three or more in G, (cd3) holds for $k = l$. If $V = U_1 \cup U_l$, then simply setting $l = 2$ completes the proof. One may thus assume that $V \supset U_1 \cup U_l$ and hence $l \geq 3$. We choose $U_{l-1}, U_{l-2}, \cdots, U_2$ in this order and show that (cd2) and (cd3) hold.

Proof

Assume for inductive hypothesis that $l \geq i + 1 \geq 3$ and the sets $U_l, U_{l-1}, \cdots, U_{i+1}$ have been appropriately chosen so that
(1) (cd2) holds for each index k, $l \geq k \geq i$, and
(2) (cd3) holds for each index k, $l \geq k \geq i + 1$.
We then show that there is a set U_i of outer vertices of G_i such that
(1) (cd2) holds for the index $k = i - 1$, and
(2) (cd3) holds for the index $k = i$.

Proof

Let $w_1, w_2, \cdots, w_t$ be the outer vertices of G_i appearing clockwise on $C_o(G_i)$ in this order, where $w_1 = v_1$ and $w_t = v_2$. There are the following two cases to consider.

Case 1: G_i is 3-connected.

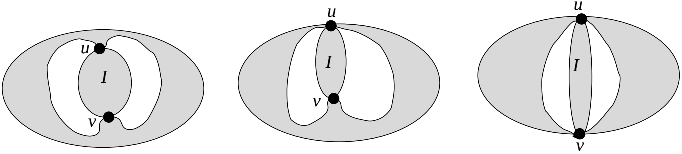
Since G_i is 3-connected and a vertex in U_{i+1} has a neighbor in G_i, there is an outer vertex $w \notin U_1$ of G_i which has a neighbor in $\overline{G_i}$. We choose the singleton set $\{w\}$ as U_i. Since G_i is 3-connected and w is an outer vertex of G_i, $G_{i-1} = G_i - w$ is internally 3-connected and w has three or more neighbors in G_{i-1}. Thus (cd2) holds for $k = i - 1$, and (cd3) holds for $k = i$.

Proof

Case 2: Otherwise.
Since $i \geq 2$, G_i is not a single cycle. G_i is internally 3-connected, but is not 3-connected. Therefore there is a chord-path for $C_o(G_i)$. Let P be a minimal chord-path for $C_o(G)$, and let w_p and w_q be the two ends of P such that $p < q$. Then $q \geq p + 2$ since G_i is internally 3-connected and $\{w_p, w_q\}$ is a separation pair of G_i. We now have the following two subcases.
Subcase 2a: $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ is an outer chain of G_i
Subcase 2b: Otherwise.

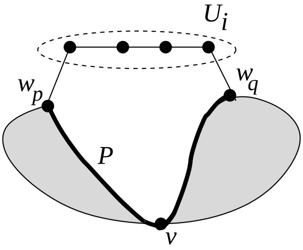
Proof

In this case we choose $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ as U_i. Since U_i is an outer chain and P is a minimal chord-path, one can observe that $U_i \cap U_1 = \emptyset$. Since G is 3-connected and each vertex $w \in U_i$ has degree two in G_i, each vertex $w \in U_i$ has a neighbor in $\overline{G_i}$ and hence (cd3) holds for $k = i$.
We now claim that G_{i-1} is internally 3-connected and hence (cd2) holds for $k = i - 1$. Assume for a contradiction that G_{i-1} is not internally 3-connected. Then G_{i-1} has either a cut vertex v or a separation pair $\{u, v\}$ having one of the three types.



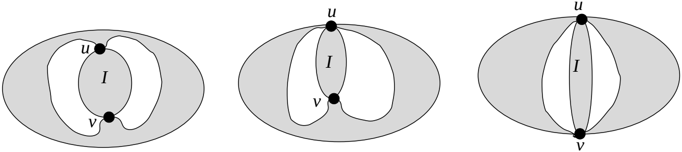
Proof

Consider first the case where G_{i-1} has a cut vertex v. Then v must be an outer vertex of G_i and $v \neq w_p, w_q$; otherwise, G_i would not be internally 3-connected. Then the minimal chord-path P above must pass through v as illustrated in Figure, contrary to the Condition (iv) of the definition of a chord-path.



Proof

Consider next the case where G_{i-1} has a separation pair $\{u, v\}$ having one of the three types.



Then $\{u, v\}$ would be a separation pair of G_i having one of the three types, and hence G_i would not be internally 3-connected, a contradiction.

Proof

Subcase 2b: Otherwise.
In this case, any vertex $w \in \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ has degree three or more in G_i; otherwise, P would not be minimal. At least one vertex $w \in \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ has a neighbor in $\overline{G_i}$; otherwise, $\{w_p, w_q\}$ would be a separating pair of G and hence G would not be 3-connected. We choose the singleton set $\{w\}$ as U_i. Then clearly $U_i \cap U_1 = \emptyset$, and (cd3) holds for $k = i$. Since w is not an end of a chord-path of $C_o(G_i)$ and G_i is internally 3-connected, $G_{i-1} = G_i - w$ is internally 3-connected and hence (cd2) holds for $k = i - 1$.

4-Canonical Decomposition

We call an ordered partition $\Pi = (U_1, U_2, \cdots, U_l)$ of set V a *4-canonical decomposition* of a plane graph $G = (V, E)$ if the following three conditions are satisfied.

(c1) U_1 consists of the two ends of an edge on $C_o(G)$, and U_l consists of the two ends of another edge on $C_o(G)$;

(c2) For each index k, $2 \leq k \leq l-1$, both G_k and $\overline{G_{k-1}}$ are 2-connected; and

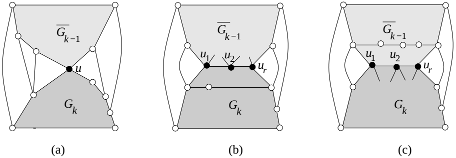
(c3) For each index k, $2 \leq k \leq l-1$, one of the following three conditions holds:

4-Canonical Decomposition

(a) U_k is a singleton set of a vertex u on $C_o(G_k)$ such that $d(u, G_k) \geq 2$ and $d(u, \overline{G_{k-1}}) \geq 2$.

(b) U_k is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) = 2$ and $d(u, \overline{G_{k-1}}) \geq 3$ for each vertex $u \in U_k$.

(c) U_k is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) \geq 3$ and $d(u, \overline{G_{k-1}}) = 2$ for each vertex $u \in U_k$.



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