Graph Classes Defined by Forbidden Subgraphs

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November 2012

Outline

Without Induced P_4

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Graphs without induced P_4s are commonly called cographs. The class of cographs is formally defined as follows.

1) A single vertex graph is a cograph.
2) If G_1 and G_2 are cographs, then $G_1 \cup G_2$ is a cographs.
3) If G is a cograph, then the complement of G is a cograph.

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- Thus the number of cographs is $2^{O(n \log n)}$.
- Adjacency testing is expensive.

Cotree

A cotree for a graph G is a tree T such that vertices correspond to leaves in the tree, internal nodes have degree at least 2 and are labeled 0 or 1, and vertices x and y of G are adjacent if and only if the least common ancestor of x and y is labeled 1.

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Applications of Cotrees

Cotrees can be used to design efficient algorithms for many problems on cographs. The fundamental technique is based on postorder traversal.

Maximum clique problem on cograph

- Construct a cotree.
- Use postorder traversal to compute the size of the maximum clique as follows.
- Mark all leaves as having maximum clique size 1.
- In the process, if we encounter a 1 node v, the maximum clique on descendants of v is equal to the sum of the maximum cliques of its children.
- If v is a 0 node, then the maximum clique on descendants of v is equal to the maximum value of the clique size among its children.

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Triangle Free Graphs

A graph is triangle-free if it does not contain the subgraph K_3.

- Number of triangle free graphs is $2^{\Theta(n^2)}$, since every bipartite graph is triangle free.

Triangle Free Graphs and Matrix Multiplication

0,1 Matrix Multiplication $A \times B$

Perform standard matrix pultiplication of $A \times B$, and replace all positive values with 1.

Triangle-free Graph Recognition

- Let A be the adjacency matrix of G.
- Compute $B =$ the result of 0,1 multiplication of $A \times A$.
- G has a triangle if and only if there is some pair of vertices x, y such that $A[x, y] = 1$ and $B[x, y] = 1$.

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Recognition of Claw-free Graphs

Trivial algorithm $O(n^4)$ time: test whether each subgraph of size 4 is a claw.

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Trivial algorithm $O(n^4)$ time: test whether each subgraph of size 4 is a claw.
Using matrix multiplication: test complement of the graph induced by the neighborhood of each vertex is triangle-free.

Hereditary Graphs

Reference: Jeremy P. Spinard, Efficient Graph Representations, American Mathematical Society, 2003.