Canonical Decomposition

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A plane graph $G$ is *internally 3-connected* if $G$ is 2-connected and, for any separation pair $\{u, v\}$ of $G$, $u$ and $v$ are outer vertices and each connected component of $G - \{u, v\}$ contains an outer vertex.

In other words, $G$ is internally 3-connected if and only if it can be extended to a 3-connected graph by adding a vertex in an outer face and connecting it to all outer vertices.
If a 2-connected plane graph $G$ is not internally 3-connected, then $G$ has a separation pair $\{u, v\}$ of one of the following three types

![Diagram showing separation pairs of different types.](image-url)
We call a path $P$ in $G$ a chord-path of the cycle $C_o(G)$ if $P$ satisfies the following (i)–(iv):

(i) $P$ connects two outer vertices $w_p$ and $w_q$, $p < q$;

(ii) $\{w_p, w_q\}$ is a separation pair of $G$;

(iii) $P$ lies on an inner face; and

(iv) $P$ does not pass through any outer edge and any outer vertex other than the ends $w_p$ and $w_q$. 
Let \( \{v_1, v_2, \cdots, v_p\} \), \( p \geq 3 \), be a set of three or more outer vertices consecutive on \( C_o(G) \) such that \( d(v_1) \geq 3 \), \( d(v_2) = d(v_3) = \cdots = d(v_{p-1}) = 2 \), and \( d(v_p) \geq 3 \). Then we call the set \( \{v_2, v_3, \cdots, v_{p-1}\} \) an outer chain of \( G \).
Let $G = (V, E)$ be a 3-connected plane graph of $n \geq 4$ vertices. For an ordered partition $\Pi = (U_1, U_2, \cdots, U_l)$ of set $V$, we denote by $G_k$, $1 \leq k \leq l$, the subgraph of $G$ induced by $U_1 \cup U_2 \cup \cdots \cup U_k$, while we denote by $\overline{G}_k$, $0 \leq k \leq l - 1$, the subgraph of $G$ induced by $U_{k+1} \cup U_{k+2} \cup \cdots \cup U_l$. Clearly $G_k = G - U_{k+1} \cup U_{k+2} \cdots \cup U_l$, and $G = G_l = \overline{G}_0$. Let $(v_1, v_2)$ be an outer edge of $G$. 
Let \((v_1, v_2)\) be an outer edge of \(G\). We then say that \(\Pi\) is a canonical decomposition of \(G\) (for an outer edge \((v_1, v_2)\)) if \(\Pi\) satisfies the following conditions (cd1)–(cd3).

**(cd1)** \(U_1\) is the set of all vertices on the inner face containing edge \((v_1, v_2)\), and \(U_l\) is a singleton set containing an outer vertex \(v_n \notin \{v_1, v_2\}\).

**(cd2)** For each index \(k, 1 \leq k \leq l\), \(G_k\) is internally 3-connected.

**(cd3)** For each index \(k, 2 \leq k \leq l\), all vertices in \(U_k\) are outer vertices of \(G_k\) and the following conditions hold:

(a) if \(|U_k| = 1\), then the vertex in \(U_k\) has two or more neighbors in \(G_{k-1}\) and has at least one neighbor in \(G_k\) when \(k < l\); and

(b) If \(|U_k| \geq 2\), then \(U_k\) is an outer chain of \(G_k\), and each vertex in \(U_k\) has at least one neighbor in \(G_k\).
Let \((v_1, v_2)\) be an outer edge of \(G\). We then say that \(\Pi\) is a canonical decomposition of \(G\) (for an outer edge \((v_1, v_2)\)) if \(\Pi\) satisfies the following conditions (cd1)–(cd3).

**(cd1)** \(U_1\) is the set of all vertices on the inner face containing edge \((v_1, v_2)\), and \(U_I\) is a singleton set containing an outer vertex \(v_n \notin \{v_1, v_2\}\).

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**(cd3)** For each index \(k, 2 \leq k \leq l\), all vertices in \(U_k\) are outer vertices of \(G_k\) and the following conditions hold:

(a) if \(|U_k| = 1\), then the vertex in \(U_k\) has two or more neighbors in \(G_{k-1}\) and has at least one neighbor in \(G_k\) when \(k < l\); and

(b) If \(|U_k| \geq 2\), then \(U_k\) is an outer chain of \(G_k\), and each vertex in \(U_k\) has at least one neighbor in \(G_k\).
Let \((v_1, v_2)\) be an outer edge of \(G\). We then say that \(\Pi\) is a \textit{canonical decomposition} of \(G\) (for an outer edge \((v_1, v_2)\)) if \(\Pi\) satisfies the following conditions (cd1)–(cd3).

\begin{enumerate}[(cd1)]
  \item \(U_1\) is the set of all vertices on the inner face containing edge \((v_1, v_2)\), and \(U_l\) is a singleton set containing an outer vertex \(v_n \notin \{v_1, v_2\}\).
  \item For each index \(k, 1 \leq k \leq l\), \(G_k\) is internally 3-connected.
  \item For each index \(k, 2 \leq k \leq l\), all vertices in \(U_k\) are outer vertices of \(G_k\) and the following conditions hold:
    \begin{enumerate}[(a)]
      \item if \(|U_k| = 1\), then the vertex in \(U_k\) has two or more neighbors in \(G_{k-1}\) and has at least one neighbor in \(\overline{G_k}\) when \(k < l\); and
      \item If \(|U_k| \geq 2\), then \(U_k\) is an outer chain of \(\overline{G_k}\), and each vertex in \(U_k\) has at least one neighbor in \(\overline{G_k}\).
    \end{enumerate}
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(a) If $|U_k| = 1$, then the vertex in $U_k$ has two or more neighbors in $G_{k-1}$ and has at least one neighbor in $\overline{G_k}$ when $k < l$; and

(b) If $|U_k| \geq 2$, then $U_k$ is an outer chain of $G_k$, and each vertex in $U_k$ has at least one neighbor in $\overline{G_k}$. 

\begin{itemize}
    \item \textbf{(a)}
    \begin{itemize}
        \item $G_{k-1}$
        \item $U_k$
    \end{itemize}
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        \item $G_{k-1}$
        \item $U_k$
    \end{itemize}
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Canonical Decomposition
Lemma Every 3-connected plane graph $G$ of $n \geq 4$ vertices has a canonical decomposition $\Pi$, and $\Pi$ can be found in linear time.
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Proof We first show that $G$ has a canonical decomposition. Let $U_1$ be the set of all vertices on the inner face containing edge $(v_1, v_2)$. Since $G$ is 3-connected and $n \geq 4$, there is an outer vertex $v_n \notin U_1$. We choose the singleton set $\{v_n\}$ as $U_l$. Thus (cd1) holds. Since $G_l = G$, (cd2) holds for $k = l$. Since $G$ is 3-connected and $v_n$ is on $C_o(G)$, $G_{l-1} = G - v_n$ is internally 3-connected and hence (cd2) holds for $k = l - 1$. Since $v_n$ has degree three or more in $G$, (cd3) holds for $k = l$. If $V = U_1 \cup U_l$, then simply setting $l = 2$ completes the proof. One may thus assume that $V \supset U_1 \cup U_l$ and hence $l \geq 3$. We choose $U_{l-1}, U_{l-2}, \ldots, U_2$ in this order and show that (cd2) and (cd3) hold.
Assume for inductive hypothesis that \( l \geq i + 1 \geq 3 \) and the sets \( U_l, U_{l-1}, \ldots, U_{i+1} \) have been appropriately chosen so that

1. (cd2) holds for each index \( k \), \( l \geq k \geq i \), and
2. (cd3) holds for each index \( k \), \( l \geq k \geq i + 1 \).

We then show that there is a set \( U_i \) of outer vertices of \( G_i \) such that

1. (cd2) holds for the index \( k = i - 1 \), and
2. (cd3) holds for the index \( k = i \).
Let \( w_1, w_2, \cdots, w_t \) be the outer vertices of \( G_i \) appearing clockwise on \( C_o(G_i) \) in this order, where \( w_1 = v_1 \) and \( w_t = v_2 \). There are the following two cases to consider.

**Case 1:** \( G_i \) is 3-connected.
Since \( G_i \) is 3-connected and a vertex in \( U_{i+1} \) has a neighbor in \( G_i \), there is an outer vertex \( w \not\in U_1 \) of \( G_i \) which has a neighbor in \( \overline{G_i} \). We choose the singleton set \( \{ w \} \) as \( U_i \). Since \( G_i \) is 3-connected and \( w \) is an outer vertex of \( G_i \), \( G_{i-1} = G_i - w \) is internally 3-connected and \( w \) has three or more neighbors in \( G_{i-1} \). Thus (cd2) holds for \( k = i - 1 \), and (cd3) holds for \( k = i \).
Case 2: Otherwise.
Since $i \geq 2$, $G_i$ is not a single cycle. $G_i$ is internally 3-connected, but is not 3-connected. Therefore there is a chord-path for $C_o(G_i)$. Let $P$ be a minimal chord-path for $C_o(G)$, and let $w_p$ and $w_q$ be the two ends of $P$ such that $p < q$. Then $q \geq p + 2$ since $G_i$ is internally 3-connected and $\{w_p, w_q\}$ is a separation pair of $G_i$. We now have the following two subcases.

Subcase 2a: $\{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\}$ is an outer chain of $G_i$

Subcase 2b: Otherwise.
In this case we choose \( \{w_{p+1}, w_{p+2}, \cdots, w_{q-1}\} \) as \( U_i \). Since \( U_i \) is an outer chain and \( P \) is a minimal chord-path, one can observe that \( U_i \cap U_1 = \emptyset \). Since \( G \) is 3-connected and each vertex \( w \in U_i \) has degree two in \( G_i \), each vertex \( w \in U_i \) has a neighbor in \( G_i \) and hence (cd3) holds for \( k = i \).

We now claim that \( G_{i-1} \) is internally 3-connected and hence (cd2) holds for \( k = i - 1 \). Assume for a contradiction that \( G_{i-1} \) is not internally 3-connected. Then \( G_{i-1} \) has either a cut vertex \( v \) or a separation pair \( \{u, v\} \) having one of the three types.
Proof

Consider first the case where $G_{i-1}$ has a cut vertex $v$. Then $v$ must be an outer vertex of $G_i$ and $v \neq w_p, w_q$; otherwise, $G_i$ would not be internally 3-connected. Then the minimal chord-path $P$ above must pass through $v$ as illustrated in Figure, contrary to the Condition (iv) of the definition of a chord-path.
Consider next the case where $G_{i-1}$ has a separation pair \{\(u, v\)\} having one of the three types.

Then \{\(u, v\)\} would be a separation pair of $G_i$ having one of the three types, and hence $G_i$ would not be internally 3-connected, a contradiction.
Subcase 2b: Otherwise.
In this case, any vertex \( w \in \{ w_{p+1}, w_{p+2}, \ldots, w_{q-1} \} \) has degree three or more in \( G_i \); otherwise, \( P \) would not be minimal. At least one vertex \( w \in \{ w_{p+1}, w_{p+2}, \ldots, w_{q-1} \} \) has a neighbor in \( \overline{G_i} \); otherwise, \( \{ w_p, w_q \} \) would be a separating pair of \( G \) and hence \( G \) would not be 3-connected. We choose the singleton set \( \{ w \} \) as \( U_i \). Then clearly \( U_i \cap U_1 = \emptyset \), and (cd3) holds for \( k = i \). Since \( w \) is not an end of a chord-path of \( C_o(G_i) \) and \( G_i \) is internally 3-connected, \( G_{i-1} = G_i - w \) is internally 3-connected and hence (cd2) holds for \( k = i - 1 \).
We call an ordered partition $\Pi = (U_1, U_2, \ldots, U_l)$ of set $V$ a 4-canonical decomposition of a plane graph $G = (V, E)$ if the following three conditions are satisfied.

\begin{itemize}
  \item[(c1)] $U_1$ consists of the two ends of an edge on $C_o(G)$, and $U_l$ consists of the two ends of another edge on $C_o(G)$;
  \item[(c2)] For each index $k$, $2 \leq k \leq l - 1$, both $G_k$ and $\overline{G_{k-1}}$ are 2-connected; and
  \item[(c3)] For each index $k$, $2 \leq k \leq l - 1$, one of the following three conditions holds:
\end{itemize}
(a) $U_k$ is a singleton set of a vertex $u$ on $C_o(G_k)$ such that $d(u, G_k) \geq 2$ and $d(u, \overline{G_{k-1}}) \geq 2$.

(b) $U_k$ is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) = 2$ and $d(u, \overline{G_{k-1}}) \geq 3$ for each vertex $u \in U_k$.

(c) $U_k$ is a set of two or more consecutive vertices on $C_o(G_k)$ such that $d(u, G_k) \geq 3$ and $d(u, \overline{G_{k-1}}) = 2$ for each vertex $u \in U_k$. 
4-Canonical Decomposition